A General Discussion on Photon Spheres in Different Categories of Spacetimes

Chen-Kai Qiao* and Ping Su†
College of Science, Chongqing University of Technology, Banan, Chongqing, 400054, China

Yang Huang‡
School of Physics and Electronic Science, Hunan University of Science and Technology, Xiangtan, Hunan, 411201, China
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Photon sphere has attracted considerable interests in the studies of black hole and other astrophysical objects. For different categories of spacetimes (or gravitational sources), the existence of photon spheres and their distributions are dramatically influenced by geometric and topological properties of spacetimes and characteristics of the corresponding gravitational fields. In this work, we carry out a geometric analysis on photon spheres for different categories of spacetime (including black hole spacetime, ultra-compact object's spacetime, regular spacetime, naked singularity spacetime). Some universal properties and conclusions are obtained for these spacetimes. We mostly focus on the existence of photon spheres, total number of photon spheres $n = n_{\text{stable}} + n_{\text{unstable}}$, the subtraction of stable photon sphere and unstable photon sphere $w = n_{\text{stable}} - n_{\text{unstable}}$ in different categories of spacetimes. These conclusions are derived solely from geometric properties of optical geometry of spacetimes, irrelevant to the specific spacetime metric forms. Besides, our results successfully recover some important theorems on photon spheres proposed in recent years.

I. INTRODUCTION

The photon spheres / circular photon orbits have attracted considerable interests in physics and astronomy since the capture of black hole images by Event Horizon Telescope (EHT) [1–3] and the gravitational wave by LIGO and Virgo [4, 5]. Firstly, photon spheres have extremely large influences on a number of astrophysical observations, such as gravitational lensing [6–9], black hole shadows [10–15], astrophysical accretion disks [15–17], quasi-normal modes [18], nonlinear stability of spacetime [19–22], and chaotic behavior of gravitational systems [18, 23]. Furthermore, photon spheres may also have the ability to reveal the underlying physical properties and characteristics of gravitational sources [24–31], providing valuable insights into gravity theories. A number of recent studies suggested that the number and distribution of photon spheres in black hole spacetimes, spacetimes produced by ultra-compact objects, regular spacetime and naked singularity spacetimes may appear entirely different features [32–36].

The prosperity of modern geometric and topology provide us new approaches and techniques on the studies of theoretical and mathematical physics, stimulating new insights and perspectives into gravitational theories. In recent years, fancy approaches using topological invariant and other mathematical concepts have emerged to explore the photon spheres / light rings in a number of gravitational systems [35–42]. In 2017, P. V. P. Cunha *et al.* proposed a topological approach to photon spheres, assigning a topological invariant / topological charge to each photon sphere /light ring using auxiliary vector fields [35]. Within this approach, they proved that the subtraction of stable and unstable photon spheres satisfies $n_{\text{stable}} - n_{\text{unstable}} = -1$ in black hole spacetimes [36], which provide a demonstration on the existence of unstable photon spheres in black hole spacetimes. Subsequently, this topological approach has been generalized into a wider range of gravitational systems and various circular orbits, including the null circular geodesics for photons and timelike circular orbits for massive particles [42–53]. Notably, S. -W. Wei *et al.* suggested that the topological charge can be used to study black hole topologies and black hole thermodynamic transitions [54–57].

Inspired by aforementioned studies, we proposed a geometric approach to study photon spheres for spherically symmetric spacetimes [58, 59]. In our approach, the photon spheres are studied in the optical geometry of 4-dimensional Lorentz spacetime. The construction of optical geometry can be viewed as a generalization of the renewed Fermat's principle into curved spacetimes [63–65]. In 2-dimensional optical geometry, two intrinsic curvatures (Gaussian curvature and geodesic curvature) completely determine the stable and unstable photon spheres. The proper location of photon spheres is uniquely constrained by geodesic curvature (via $\kappa_g(r_{ph}) = 0$), and the sign of Gaussian curvature determines the stability of photon spheres. Particularly, the negative Gaussian curvature suggests that the photon sphere is unstable, while the positive Gaussian curvature indicates the photon sphere is stable. Furthermore, it is proved that our geometric approach obtains completely equivalent results with the conventional approach (using effective potentials) and the topological approach on photon spheres [58–61]. The geometric viewpoints behind our geometry approach have stimulated several studies in related fields, such as the extension into timelike circular geodesics and particle surfaces (in which the curvatures in Jacobi geometry of spacetimes are utilized) [61, 62].

* Email: chenkaiqiao@cqut.edu.cn
† Email: suping@cqut.edu.cn
‡ Email: yanghuang@mail.bnu.edu.cn

The existence of photon spheres and their distribution features are important topics. The classical Schwarzschild, Reissner-Nordstrom(RN), Kerr black holes admit a single unstable photon sphere. On the other hand, hairy black hole solutions coupled with scalar matter fields and electromagnetic fields could admit multiple photon spheres [67–70]. For black hole systems, because of the capture of shadow images in astrophysical observations, it is widely accepted that the existence of unstable photon sphere must be a general feature in black hole spacetimes. Recently, a number of scholars proved the existence of unstable photon spheres in the vicinity of black holes, based on several different methods [36, 42, 71–77]. However, in other categories of spacetime (such as ultra compact object's spacetime, regular spacetime, naked singularity spacetime), the existence of photon spheres may become subtle. For instance, the studies on naked singularity spacetime shown that the unstable photon spheres can disappear in such spacetimes [78, 79]. Moreover, the presence of photon spheres in regular spacetime are more complex than conventional black hole spacetime, which could exhibit interesting behaviors [32, 33, 80]. Particularly, the presence and absence of event horizon have strong influences on the number of photon spheres in different kind of spacetimes [35]. Under such circumstances, a general and comprehensive discussion on photon spheres in different categories of spacetime is necessary and important, which may provide a pathway to distinguish different kinds of spacetime theoretically (or observationally) and deepen our understanding on the properties of such spacetimes.

In the current work, we carry out a comprehensive analysis on photon spheres in several categories of spacetimes (including black hole spacetime, ultra-compact object's spacetime, regular spacetime, naked singularity spacetime), using our geometric approach in references [58–60]. Assuming the most common asymptotic behaviors of spacetime (the asymptotically flat, asymptotically de-Sitter and asymptotically anti de-Sitter), the existence of photon spheres, total number of photon spheres $n = n_{\text{stable}} + n_{\text{unstable}}$, the distribution features of stable and unstable photon spheres, and the subtraction of stable photon spheres and unstable photon spheres $w = n_{\text{stable}} - n_{\text{unstable}}$ (which can be viewed as a topological charge / topological invariant in different kinds of spacetime), are studied and discussed in details for these categories of spacetimes. This analysis would enable us to extract some universal and common features on photon spheres in different categories of spacetimes. Furthermore, it should be mentioned that our geometric analysis in the presented work is suitable to general (static) spherically symmetric spacetimes, with spacetime metric to be $ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2$, irrespective of any specific forms of spacetime metric components.

This rest of the present work is organized as follows. Section II concisely reviews our geometric approach to photon spheres. Section III presents the discussions on the existence of photon spheres in different categories of spacetimes. Section IV studies the distribution properties of stable and unstable photon spheres, especially the subtraction of stable and unstable photon spheres $w = n_{\text{stable}} - n_{\text{unstable}}$. The conclusions and perspectives are summarized in section V.

II. GEOMETRIC APPROACH TO PHOTON SPHERES

In this section, we give a concise review on the geometric approach to photon spheres, which was proposed in recent works [58–60]. In this approach, the low-dimensional analog of Lorentz spacetime — the optical geometry — is employed to analyze photon spheres. In spherically symmetric gravitational systems, two intrinsic curvatures (geodesic curvature and Gaussian curvature) in optical geometry are capable to determine the locations of photon spheres and their stability.

Mathematically, the optical geometry can be constructed in several equivalent ways [63–66]. A simple approach to realize the optical geometry is from a continuous map of the spacetime geometry using the null constraint $d\tau^2 = -ds^2 = 0$

$$\underbrace{ds^2 = g_{\mu\nu}dx^\mu dx^\nu}_{\text{Spacetime Geometry}} \overset{d\tau^2=-ds^2=0}{\Longrightarrow} \underbrace{dt^2 = g_{ij}^{\text{OP}}dx^i dx^j}_{\text{Optical Geometry}} \tag{1}$$

For any spherically symmetric spacetimes, the photon orbits can always be constricted in the equatorial plane without loss of generality. In such circumstances, a 2-dimensional optical geometry can be constructed.

$$\underbrace{dt^2 = g_{ij}^{\text{OP}}dx^i dx^j}_{\text{Optical Geometry}} \overset{\theta=\pi/2}{\Longrightarrow} \underbrace{dt^2 = \tilde{g}_{ij}^{\text{OP-2d}}dx^i dx^j}_{\text{Optical Geometry (Two Dimensional)}} \tag{2}$$

Particularly, in spherically symmetric spacetimes, the corresponding optical geometry $dt^2 = \tilde{g}_{ij}^{\text{OP-2d}}dx^i dx^j$ is always described by 2-dimensional Riemannian geometry [64, 65, 81–86].

In our geometric approach, the analysis of photon spheres and their stability is carried out using intrinsic curvatures in 2-dimensional optical geometry. The most important intrinsic curvatures in 2-dimensional optical geometry are Gaussian curvature and geodesic curvature. Based on these intrinsic curvatures, the locations of photon spheres and their stability can be completely determined. Firstly, any photon spheres could maintain their geodesic nature when transformed in the 2-dimensional optical geometry, so the geodesic curvature for photon sphere vanishes naturally [58, 59]

$$\kappa_g(r = r_{ph}) = 0 \tag{3}$$

Further, it has been proven that the geodesic curvature condition in our geometric approach is totally equivalent

TABLE I. Features of the conventional effective potential approach, the topological approach, and our geometric approach.

Approach	Our Geometric Approach	Topological Approach	Conventional Approach
Geometry	Optical Geometry	Spacetime Geometry	Spacetime Geometry
Basic quantities	Gaussian Curvature $\mathcal{K}(r)$ Geodesic Curvature $\kappa_g(r)$	Auxiliary Vector Field V Topological Charge w	Effective Potential $V_{\text{eff}}(r)$
Photon Sphere	$\kappa_g(r)=0$	$V=0$	$\frac{dV_{\text{eff}}(r)}{dr}=0$
Unstable Photon Sphere	$\kappa_g(r)=0$ and $\mathcal{K}(r)<0$	$V=0$ and $w=-1$	$\frac{dV_{\text{eff}}(r)}{dr}=0$ and $\frac{d^2V_{\text{eff}}(r)}{dr^2}<0$
Stable Photon Sphere	$\kappa_g(r)=0$ and $\mathcal{K}(r)>0$	$V=0$ and $w=+1$	$\frac{dV_{\text{eff}}(r)}{dr}=0$ and $\frac{d^2V_{\text{eff}}(r)}{dr^2}>0$

to the effective potential condition in conventional approach [58, 59].

$$\kappa_g(r=r_{\text{ph}})=0 \quad\Leftrightarrow\quad \left.\frac{dV_{\text{eff}}(r)}{dr}\right|_{r=r_{\text{ph}}}=0 \tag{4}$$

Secondly, the stability of photon spheres can be constrained by conjugate points in optical geometry. For stable and unstable photon spheres, the behavior of photon orbits undergo a perturbation are completely different. When photon orbits get perturbed from unstable photon spheres, they would move far away and never return to unstable photon spheres. Conversely, when photon orbits are perturbed from stable photon spheres, they could also form another bound orbits near stable photon spheres. Mathematically, these distinct features of stable and unstable photon sphere are reflected by conjugate points in the manifold. There are conjugate points in the stable photon sphere, while no conjugate points exist in the unstable photon sphere [58–60]. The presence and absence of conjugate points provide us a novel scheme to distinguish the stable and unstable photon spheres. The Cartan-Hadamard theorem in differential geometry and topology strongly constrain the Gaussian curvature and the existence of conjugate points [87]. Applying the Cartan-Hadamard theorem in the optical geometry, the following Gaussian curvature condition on the stability of photon spheres is derived [58, 59].

$$\begin{aligned}&\mathcal{K}(r)<0 \;\Rightarrow\; \text{The photon sphere } r=r_{ph} \text{ is unstable}\\ &\mathcal{K}(r)>0 \;\Rightarrow\; \text{The photon sphere } r=r_{ph} \text{ is stable}\end{aligned}$$

This Gaussian curvature condition, derived purely from the curvatures and topology of optical geometry, is independent of the specific metric forms of the particular gravitational systems. Consequently, the conclusion can be universally applied to any spherically spherical systems. Furthermore, this Gaussian curvature condition for stable and unstable photon spheres is equivalent to the effective potential condition in conventional approach [58, 59]

$$\left.\frac{d^2V_{\text{eff}}(r)}{dr^2}\right|_{r=r_{\text{unstable}}}<0 \quad\Leftrightarrow\quad \mathcal{K}(r=r_{\text{unstable}})<0 \tag{5a}$$

$$\left.\frac{d^2V_{\text{eff}}(r)}{dr^2}\right|_{r=r_{\text{stable}}}>0 \quad\Leftrightarrow\quad \mathcal{K}(r=r_{\text{stable}})>0 \tag{5b}$$

The distinguishing features of our geometric approach, conventional approach, topological approach and their equivalence have been summarized in table I.

III. THE EXISTENCE OF PHOTON SPHERES

In this section, we present a discussion on the existence of photon spheres in general spherically symmetric gravitational systems, including black hole spacetimes, ultra-compact objects' spacetimes, regular spacetimes (free of spacetime singularities), and naked singularity spacetimes.

The general (static) spherically symmetric spacetime metric can be expressed as

$$ds^2=g_{tt}(r)dt^2+g_{rr}(r)dr^2+g_{\theta\theta}(r)d\theta^2+g_{\phi\phi}(r,\theta)d\phi^2 \tag{6}$$

Additionally, for any spherically symmetric spacetimes, the metric components $g_{\theta\theta}=r^2$ and $g_{\phi\phi}=r^2\sin^2\theta$ can always be achieved through a coordinate transformation, which makes the spacetime metric reduced to be

$$ds^2=-f(r)dt^2+g(r)dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2 \tag{7}$$

with $f(r)=-g_{tt}$ and $g(r)=g_{rr}$. The corresponding 2-dimensional optical geometry (restricted in the equatorial plane $\theta=\pi/2$) of spherically symmetric spacetime is given by

$$dt^2=\tilde{g}_{ij}^{\text{OP-2d}}dx^idx^j=\frac{g(r)}{f(r)}\cdot dr^2+\frac{r^2}{f(r)}\cdot d\phi^2 \tag{8}$$

The geodesic curvature of a circular curve with constant radius r (such as photon spheres) in this 2-dimensional optical geometry can be calculated via

$$\begin{aligned}\kappa_g(r) &= \frac{1}{2\sqrt{\tilde{g}_{rr}^{\text{OP-2d}}}}\frac{\partial\left[\log(\tilde{g}_{\phi\phi}^{\text{OP-2d}})\right]}{\partial r}\\ &= \frac{1}{\sqrt{f(r)\cdot g(r)}}\left[\frac{f(r)}{r}-\frac{1}{2}\frac{df(r)}{dr}\right]\end{aligned} \tag{9}$$

and the Gaussian curvature in 2-dimensional optical geometry can be calculated through

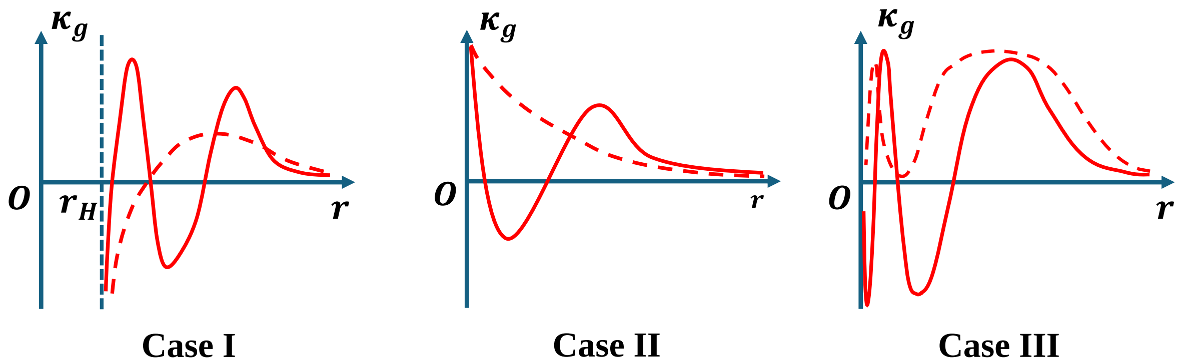

FIG. 1. The variation of geodesic curvature $\kappa_g(r)$ with respect to radial coordinate r in different category of spacetimes. **Case I:** For black hole spacetime and regular spacetime (with the presence of horizons) satisfying $\lim_{r\to r_H}\kappa_g(r)<0$ and $\lim_{r\to\infty}\kappa_g(r)=0^+$, the equation $\kappa_g(r)=0$ must have at least one solutions outside the event horizon. The dashed line draws the one-solution case, and the solid line draws the three-solution case. **Case II:** For ultra-compact object's spacetime, naked singularity spacetime (with finite first order metric derivative), and regular spacetime (without horizons) satisfying $\lim_{r\to 0}\kappa_g(r)>0$ and $\lim_{r\to\infty}\kappa_g(r)=0^+$, the equation $\kappa_g(r)=0$ must have either no solution or an even number of solutions. The dashed line illustrates the no-solution case, and the solid line draws the two-solution case. **Case III:** For naked singularity spacetime (with first order metric derivative diverge to positive infinity at spacetime singularity $\lim_{r\to 0}\frac{df(r)}{dr}=+\infty$), the geodesic curvature satisfies $\lim_{r\to 0}\kappa_g(r)=$ indefinite and $\lim_{r\to\infty}\kappa_g(r)=0^+$. It is impossible to determine whether equation $\kappa_g(r)=0$ has solutions or not. The solid line depicts an example where the solution of $\kappa_g(r)=0$ exists, while the dashed line illustrates an example where $\kappa_g(r)=0$ has no solutions. In this figure, the notation $\lim_{r\to\infty}\kappa_g(r)=0^+$ represents that the geodesic curvature satisfies $\kappa_g(r)\to 0$ and $\kappa_g(r)>0$ in the infinite distance limit $r\to\infty$.

$$\begin{aligned}\mathcal{K} &= -\frac{1}{\sqrt{\tilde{g}^{\text{OP-2d}}}}\left[\frac{\partial}{\partial\phi}\left(\frac{1}{\sqrt{\tilde{g}^{\text{OP-2d}}_{\phi\phi}}}\frac{\partial\sqrt{\tilde{g}^{\text{OP-2d}}_{rr}}}{\partial\phi}\right)+\frac{\partial}{\partial r}\left(\frac{1}{\sqrt{\tilde{g}^{\text{OP-2d}}_{rr}}}\frac{\partial\sqrt{\tilde{g}^{\text{OP-2d}}_{\phi\phi}}}{\partial r}\right)\right]\\ &= \frac{1}{2r}\frac{1}{g(r)}\frac{df(r)}{dr}+\frac{1}{2r}\frac{f(r)}{[g(r)]^2}\frac{dg(r)}{dr}-\frac{1}{2f(r)g(r)}\left[\frac{df(r)}{dr}\right]^2-\frac{1}{4[g(r)]^2}\frac{df(r)}{dr}\frac{dg(r)}{dr}+\frac{1}{2g(r)}\frac{d^2f(r)}{dr^2}\end{aligned} \tag{10}$$

with $\tilde{g}^{\text{OP-2d}}=\det\left(\tilde{g}^{\text{OP-2d}}_{ij}\right)$ to be the determinant of 2-dimensional optical geometry metric. In the present work, all conclusions on photon sphere are obtained from geometric analysis (using geodesic curvature in optical geometry) and the asymptotic behaviors of gravitational systems, irrespective of any specific spacetime metric functions g_{tt}, g_{rr}, $g_{\theta\theta}$ and $g_{\phi\phi}$.

From the geometric approach described in section II, the condition for photon spheres is the vanishing of geodesic curvature, namely $\kappa_g(r_{ph})=0$. Therefore, the existence of photon spheres suggest that equation $\kappa_g(r)=0$ admits at least one solution [60]. To provide an analysis on the existence of photon spheres in general spherically symmetric gravitational systems, the crucial point is studying the behavior of geodesic curvature $\kappa_g(r)$ for a circular curve in the 2-dimensional optical geometry.

For different categories of spacetime, the valid regions in which photon spheres could appear are slightly different. When a spacetime has an event horizon, the valid region for the appearance of photon spheres is generally outside the horizon $r>r_H$ (the photon spheres inside the horizons, if existed, are unable to be captured by any astrophysical observations and observer outside the horizon). On the contrary, if a spacetime does not have event horizons, the positions of photon sphere may occur in the entire spacetime region $r>0$. To give a comprehensive discussion on the existence of photon spheres, the geodesic curvature $\kappa_g(r)$ in the inner and outer boundaries of the valid region (where photon spheres could exist) needed to be studied for different categories of spacetimes. Particularly, for black hole spacetime and regular spacetime (without singularities, but have horizons), the geodesic curvature in the near horizon region $r\to r_H$ and the infinite distance region $r\to\infty$ should be analyzed. For ultra-compact object's spacetime, naked singularities and regular spacetime (without singularity spacetime and horizons), the geodesic curvature in the $r\to 0$ limit and the infinite distance region $r\to\infty$ must be analyzed. In

the rest of present work, we shall encounter the following situations:

- **CASE I**: For black hole spacetime and regular spacetime with the presence of horizons, the geodesic curvature in the infinite distance limit satisfies $\lim_{r\to\infty}\kappa_g(r)=0^+$ and in near horizon region satisfies $\lim_{r\to r_H}\kappa_g(r)<0$, the equation $\kappa_g(r)=0$ must have at least one solution outside the event horizon. This situation is illustrated in the first case of figure 1.

- **CASE II:** For ultra-compact object's spacetime (without horizons), naked singularity spacetime (with finite first order metric derivative), regular spacetime (without horizons), the geodesic curvature in the infinite distance limit satisfies $\lim_{r\to\infty}\kappa_g(r)=0^+$ and in center limit satisfies $\lim_{r\to 0}\kappa_g(r)>0$, the equation $\kappa_g(r)=0$ must have either no solution or an even number of solutions. This situation is shown in the second case of figure 1.

- **CASE III**: For naked singularity spacetime (with first order metric derivative diverge to positive infinity at spacetime singularity $\lim_{r\to 0}\frac{df(r)}{dr}=+\infty$), the geodesic curvature in the infinite distance limit satisfies $\lim_{r\to\infty}\kappa_g(r)=0^+$, while the geodesic curvature in the center limit becomes indefinite ($\lim_{r\to 0}\kappa_g(r)=$ indefinite). Under this circumstance, it is impossible to determine whether equation $\kappa_g(r)=0$ has solutions or not. This situation has been shown in the third case of figure 1.

In the rest of this section, we analyze the behavior of geodesic curvature $\kappa_g(r)$ for each cases of figure 1 in different category of spacetimes independently.

A. Geodesic Curvature in the Infinite Distance Limit $r\to\infty$

This subsection discusses the behavior of geodesic curvature $\kappa_g(r)$ in the infinite distance region $r\to\infty$ in the 2-dimensional optical geometry. As we will see in the following procedures, the geodesic curvature in the infinite distance limit only depends on the asymptotic expansions of spacetime metric, irrelevant to the presence (or absence) of event horizon. The analysis and conclusion are completely the same for different categories of spacetimes (such as black hole spacetimes, ultra-compact objects' spacetimes, regular spacetimes, naked singularity spacetimes). In particular, three most common asymptotic behaviors of spacetime (the asymptotically flat, asymptotically de-Sitter and asymptotically anti de-Sitter) are chosen to carry out the analysis.

Asymptotically Flat Spacetimes: The asymptotically flat spacetime has the following asymptotic metric expansions in the infinite distance limit $r\to\infty$

$$\lim_{r\to\infty} f(r)=1+\frac{a_i}{r^i}+O\left(\frac{1}{r^{i+1}}\right)\qquad (i\geq 1)\tag{11a}$$

$$\lim_{r\to\infty} g(r)=\frac{1}{1+\frac{b_j}{r^j}+O\left(\frac{1}{r^{j+1}}\right)}\qquad (j\geq 1)\tag{11b}$$

with a_i and b_j to be expansion coefficients. Based on asymptotic metric expansions, the geodesic curvature $\kappa_g(r)$ in the infinite distance limit is calculated as

$$\begin{aligned}\lim_{r\to\infty}\kappa_g(r) &= \lim_{r\to\infty}\left[\frac{1}{\sqrt{f(r)\cdot g(r)}}\left(\frac{f(r)}{r}-\frac{1}{2}\frac{df(r)}{dr}\right)\right]\\ &= \lim_{r\to\infty}\frac{\frac{1}{r}+\frac{a_i}{r^{i+1}}+\frac{ia_i}{2r^{i+1}}+O\left(\frac{1}{r^{i+2}}\right)}{\sqrt{1+\frac{\frac{a_i}{r^i}-\frac{b_j}{r^j}+O\left(\frac{1}{r^{i+1}},\frac{1}{r^{j+1}}\right)}{1+\frac{b_j}{r^j}+O\left(\frac{1}{r^{j+1}}\right)}}}\\ &= \lim_{r\to\infty}\frac{1}{r}=0^+\end{aligned}\tag{12}$$

The notation $\lim_{r\to\infty}\kappa_g(r)=0^+$ represents that the geodesic curvature satisfies $\kappa_g(r)\to 0$ and $\kappa_g(r)>0$ in the $r\to\infty$ limit. This is exactly consistent with the three cases illustrated in figure 1.

Asymptotically de Sitter and Asymptotically Anti de-Sitter Spacetimes: In the infinite distance limit, the asymptotically de Sitter / asymptotically anti de-Sitter spacetime has the asymptotic analytical metric expansions

$$\lim_{r\to\infty} f(r)=1-\frac{\Lambda r^2}{3}+\frac{a_i}{r^i}+O\left(\frac{1}{r^{i+1}}\right)\qquad (i\geq 1)\tag{13a}$$

$$\lim_{r\to\infty} g(r)=\frac{1}{1-\frac{\Lambda r^2}{3}+\frac{b_j}{r^j}+O\left(\frac{1}{r^{j+1}}\right)}\qquad (j\geq 1)\tag{13b}$$

with a_i and b_j to be expansion coefficients and Λ to be the cosmological constant. The asymptotically de Sitter / anti de Sitter spacetime admits a positive (or negative) cosmological constant $\Lambda>0$ (or $\Lambda<0$). Using the above metric expansions, we obtain the geodesic curvature in the infinite distance limit

$$\begin{aligned}\lim_{r\to\infty}\kappa_g(r) &= \lim_{r\to\infty}\left[\frac{1}{\sqrt{f(r)\cdot g(r)}}\left(\frac{f(r)}{r}-\frac{1}{2}\frac{df(r)}{dr}\right)\right]\\ &= \lim_{r\to\infty}\frac{\frac{1}{r}+\frac{a_i}{r^{i+1}}+\frac{ia_i}{2r^{i+1}}+O\left(\frac{1}{r^{i+2}}\right)}{\sqrt{1+\frac{\frac{a_i}{r^i}-\frac{b_j}{r^j}+O\left(\frac{1}{r^{i+1}},\frac{1}{r^{j+1}}\right)}{1-\frac{\Lambda r^2}{3}+\frac{b_j}{r^j}+O\left(\frac{1}{r^{j+1}}\right)}}}\\ &= \lim_{r\to\infty}\frac{1}{r}=0^+\end{aligned}\tag{14}$$

The notation $\lim_{r\to\infty}\kappa_g(r)=0^+$ indicates that the geodesic curvature satisfies $\kappa_g(r)\to 0$ and $\kappa_g(r)>0$ in the $r\to\infty$ limit, which is the same with results for asymptotically flat spacetimes. The geodesic curvature behavior in the infinite distance limit successfully agrees with the three cases illustrated in figure 1.

B. Geodesic Curvature in the Near Horizon Limit $r \to r_H$ (or in the Center Limit $r \to 0$)

Unlike the behavior of geodesic curvature in the infinite distance limit (which depends uniquely on the asymptotic analytical expansions of metric components and is irrelevant to the horizons and types of spacetime), the geodesic curvature in the inner boundary of region (where photon sphere could exist) are strongly constrained by spacetime structures and properties of gravitational sources. For different categories of spacetimes (the black hole spacetime, ultra-compact object's spacetime, regular spacetime, naked singularity spacetime), the geodesic curvature $\kappa_g(r)$ in the inner boundary are classified into two situations: the near horizon limit $r \to r_H$ and the center limit $r \to 0$.

Black Hole Spacetime: According to the cosmic censorship conjecture proposed by R. Penrose *et al.*, black hole spacetimes always admit event horizons, such that spacetime singularities are enclosed by event horizons [88–90]. The event horizon is a null hypersurface in Lorentz spacetime and vanishes the metric component $g_{tt} = -f(r)$, $g_{rr} = g(r)$, and it plays significantly important roles in the causal structure of spacetime. The metric components $f(r)$ and $g(r)$ must change their signs when going across the horizon. For astronomical observational reason, the locations of photon sphere are always assumed to be outside the black hole horizon, since any photon spheres inside event horizon will never be observed by a distant observer.

Notably, for black hole spacetime, a recent study have shown that the geodesic curvature in the near horizon limit is closely related to the surface gravity of black holes, via $\lim_{r\to r_H} \kappa_g(r) = -\kappa_{\text{surface gravity}} < 0$ [61]. Since the geodesic curvature in the infinite distance limit satisfies $\lim_{r\to\infty} \kappa_g(r) = 0^+$ (which have been discussed in subsection III A), then the equation $\kappa_g(r) = 0$ must have at least one solutions (as illustrated in the first case of figure 1), confirming the existence of photon spheres in the vicinity of black holes.

However, in the present work, we apply an alternative demonstration on the existence of photon spheres in black hole spacetime, through a simpler analysis on metric functions $g_{tt} = -f(r)$ and $g_{rr} = g(r)$ and their derivatives near the event horizon. Since the presence of event horizon makes the change of sign for metric components $f(r)$ and $g(r)$, the metric component function near the event horizon must obey the relations

$$\begin{aligned} f(r) < 0 &\quad \text{when } r < r_H \\ f(r) = 0 &\quad \text{when } r \to r_H \\ f(r) = \text{finite} > 0 &\quad \text{when } r > r_H \end{aligned}$$

which suggests in near horizon limit

$$\lim_{r\to r_H} \frac{df(r)}{dr} > 0 \qquad \lim_{r\to r_H} \frac{f(r)}{r} = 0$$

With these relations, the geodesic curvature in 2-dimensional optical geometry in the near horizon limit becomes

$$\begin{aligned} \lim_{r\to r_H} \kappa_g(r) &= \lim_{r\to r_H} \left[\frac{1}{\sqrt{f(r)\cdot g(r)}}\left(\frac{f(r)}{r} - \frac{1}{2}\frac{df(r)}{dr}\right)\right] \\ &= \lim_{r\to r_H} \left[\frac{1}{\sqrt{f(r)\cdot g(r)}}\left(0 - \frac{1}{2}\frac{df(r)}{dr}\right)\right] \\ &< 0 \end{aligned} \tag{15}$$

which is consistent with the result obtained by Cunha *et al.* in reference [61]. In the above derivation process, it should be noted that the factor $\frac{1}{\sqrt{f(r)\cdot g(r)}}$ is assumed to be non-singular in the near horizon limit. Particularly, in the simple spherically symmetric black hole systems (such as the classical Schwarzschild black hole and RN black hole), the metric components satisfy $f(r)\cdot g(r) = 1$. In the most general spherically symmetric black hole systems, although the relationship $f(r)\cdot g(r) = 1$ is no longer valid, it is reasonable to assume that the product $f(r)\cdot g(r) > 0$ holds outside the event horizon. This can be viewed as a restriction from the causal structure of spacetime, such that the tangent vectors $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial r}$ associated with the conventional static coordinates point to the timelike and spacelike directions simultaneously outside the event horizon.

In conclusion, for black hole spacetime, the geodesic curvature in optical geometry satisfies $\lim_{r\to r_H} \kappa_g(r) < 0$ in the near horizon limit and $\lim_{r\to\infty} \kappa_g(r) = 0^+$ in the infinite distance limit, as illustrated in the first case of figure 1. The existence of solution for equation $\kappa_g(r) = 0$ is guaranteed in such cases, demonstrating the presence of photon spheres near black holes. Furthermore, assuming the continuity of geodesic curvature $\kappa_g(r)$ [92], we then observe that the total number of photon spheres for black hole spacetime (outside the event horizon) must be an odd number, namely $n = n_{\text{stable}} + n_{\text{unstable}} = 2k + 1$ (where $k \in \mathbb{N}$ is a natural number).

Ultra-Compact Object's Spacetime: For spacetimes generated by ultra-compact objects (such as neutron stars, dwarf stars and other massive compact objects), due to the absence of an event horizon, the photon sphere may emerge in the entire spacetime region $r \geq 0$ [91]. For such spacetime, it is necessary to analyze the behavior of geodesic curvature $\kappa_g(r)$ in the center limit $r \to 0$.

The ultra-compact objects do not produce event horizons, so the spacetime metric is regular in the entire spacetime

$$\begin{aligned} f(r) = \text{finite} > 0 &\quad \text{when } r \geq 0 \\ g(r) = \text{finite} > 0 &\quad \text{when } r \geq 0 \\ \frac{df(r)}{dr} = \text{finite} &\quad \text{when } r \geq 0 \end{aligned}$$

which suggests in the center limit $r \to 0$

$$\lim_{r\to 0} \frac{f(r)}{r} = +\infty$$

In such cases, the geodesic curvature satisfies the following relation in the $r \to 0$ limit

$$\begin{aligned}\lim_{r\to 0}\kappa_g(r) &= \lim_{r\to 0}\left[\frac{1}{\sqrt{f(r)\cdot g(r)}}\left(\frac{f(r)}{r}-\frac{1}{2}\frac{df(r)}{dr}\right)\right]\\ &= \lim_{r\to 0}\left[\frac{1}{\sqrt{f(r)\cdot g(r)}}\Big(+\infty-\text{finite}\Big)\right.\\ &= +\infty \end{aligned} \tag{16}$$

Therefore, for spacetimes generated by ultra-compact objects, the geodesic curvature satisfy $\lim_{r\to 0}\kappa_g(r) = +\infty$ in the center limit and $\lim_{r\to\infty}\kappa_g(r) = 0^+$ in the infinite distance limit (the latter has been explained in subsection III A), this is precisely the scenario depicted in the second case of figure 1. In such cases, the equation $\kappa_g(r) = 0$ possesses either no solution or an even number of solutions (we have assumed the continuity of geodesic curvature, as explained in footnote [92]). Consequently, the existence of photon spheres is not necessary for ultra-compact object's spacetimes (such as spacetime generated by neutron stars, dwarf stars and other massive compact objects). Furthermore, for ultra compact object's spacetime, the total number of photon sphere must be even, namely $n = n_{\text{stable}} + n_{\text{unstable}} = 2k$ (where $k \in \mathbb{N}$ is a natural number).

Regular Spacetime (with the presence of event horizon): In this part, we analyze the regular spacetimes, which are free of spacetime singularities but with the presence of event horizons. Similar to the black hole cases, we only study photon spheres that located outside the event horizon for astronomical observational reason (photon spheres inside event horizon will not be observed by a distant observer). Although a recent study shows that photon spheres maybe exist inside the event horizon for special regular spacetime [32], their connections with observations is still greatly challenged. Therefore, we only focus on photon spheres outside the event horizons, the discussion of photon spheres inside horizon is beyond the scope of our present study.

For regular spacetime that possesses a horizon, since the event horizon changes the sign of metric components $f(r)$ and $g(r)$, the metric components near and outside the horizon must obey the relation

$$\begin{aligned}\lim_{r\to r_H} f(r) = 0 &\quad \text{when } r \to r_H\\ f(r) = \text{finite} > 0 &\quad \text{when } r > r_H\\ g(r) = \text{finite} > 0 &\quad \text{when } r > r_H\end{aligned}$$

which suggests in near horizon limit $r \to r_H$

$$\lim_{r\to r_H}\frac{df(r)}{dr} > 0 \qquad \lim_{r\to r_H}\frac{f(r)}{r} = 0$$

Then the geodesic curvature in optical geometry in the near horizon limit becomes

$$\begin{aligned}\lim_{r\to r_H}\kappa_g(r) &= \lim_{r\to r_H}\left[\frac{1}{\sqrt{f(r)\cdot g(r)}}\left(\frac{f(r)}{r}-\frac{1}{2}\frac{df(r)}{dr}\right)\right]\\ &= \lim_{r\to r_H}\left[\frac{1}{\sqrt{f(r)\cdot g(r)}}\left(0-\frac{1}{2}\frac{df(r)}{dr}\right)\right]\\ &< 0 \end{aligned} \tag{17}$$

Similar to the black hole spacetime, here we have assume the factor $\frac{1}{\sqrt{f(r)\cdot g(r)}}$ to be non singular in the near horizon limit (such that $f(r)\cdot g(r) > 0$ is always hold outside the event horizon). In conclusion, for regular spacetime with event horizons, the geodesic curvature satisfies $\lim_{r\to r_H}\kappa_g(r) < 0$ in the near horizon limit and $\lim_{r\to\infty}\kappa_g(r) = 0^+$ at infinite distance limit (the latter has been discussed in subsection III A), which is the same as the first case of figure 1. Under these circumstances, the existence of solutions in equation $\kappa_g(r)$ is guaranteed, providing the explicit substantiation for the presence of photon spheres in such regular spacetimes. Furthermore, assuming the continuity of geodesic curvature, it can be clearly demonstrated that the total number of photon spheres around regular spacetimes (with event horizon) must be an odd number $n = n_{\text{stable}} + n_{\text{unstable}} = 2k + 1$ (where $k \in \mathbb{N}$ is a natural number).

Regular Spacetime (without the presence of event horizon): Opposite to the previous part (where the regular spacetime has event horizon), for the regular spacetime without horizons and singularities simultaneously, the photon sphere may emerge in the entire region of spacetime $r \geq 0$. It is necessary to analyze the behavior of geodesic curvature in the center limit $r \to 0$.

For the regular spacetime without a horizon, the metric components $f(r)$ and $g(r)$ must be regular and do not change their sign, namely

$$\begin{aligned}f(r) = \text{finite} > 0 &\quad \text{when } r \geq 0\\ g(r) = \text{finite} > 0 &\quad \text{when } r \geq 0\end{aligned}$$

which suggests $\lim_{r\to 0}\frac{f(r)}{r} = +\infty$ in the center limit. Meanwhile, regular spacetimes do not admit any spacetime singularities, so the derivative of metric component must be finite in the entire spacetime

$$\frac{df(r)}{dr} = \text{finite} \quad \text{when } r \geq 0$$

Therefore, geodesic curvature in the center limit reduces to

$$\begin{aligned}\lim_{r\to 0}\kappa_g(r) &= \lim_{r\to 0}\left[\frac{1}{\sqrt{f(r)\cdot g(r)}}\left(\frac{f(r)}{r}-\frac{1}{2}\frac{df(r)}{dr}\right)\right]\\ &= \lim_{r\to 0}\left[\frac{1}{\sqrt{f(r)\cdot g(r)}}\Big(+\infty-\text{finite}\Big)\right.\\ &= +\infty \end{aligned} \tag{18}$$

Combining the geodesic curvature $\lim_{r\to\infty}\kappa_g(r) = 0^+$ at infinite distance limit and $\lim_{r\to 0}\kappa_g(r) = +\infty$ in the center limit, this scenario is precisely what we depicted in

the second case of figure 1, which implies that the equation $\kappa_g(r) = 0$ possesses either no solution or even number of solutions. Consequently, the existence of photon spheres is not necessary for regular spacetimes without event horizons. And the total number of photon spheres is characterized by $n = n_{\text{stable}} + n_{\text{unstable}} = 2k$, where $k \in \mathbb{N}$ is a natural number.

Naked Singularity Spacetime (with Finite First Order Derivatives of Metric): The most extraordinary characteristic of naked singularity spacetime is the emergence of naked singularity that can be seen by a distant observer, which is not enclosed by any event horizons. For spherically symmetric naked singularity spacetimes discussed in this work, we restrict the naked singularity in the center position $r = 0$. We first consider the situation where the first order derivatives of metric components are finite in the entire spacetime, which enable the proper definition of Riemannian curvature tensors at spacetime singularity position $r = 0$ (although the geometric invariants constructed using Riemannian curvature tensors must be divergent at spacetime singularity).

The naked singularity spacetime do not have event horizons, thus the signs of metric components $f(r)$ and $g(r)$ are kept unchanged

$$\begin{aligned} f(r) &= \text{finite} > 0 \quad \text{when } r \geq 0 \\ g(r) &= \text{finite} > 0 \quad \text{when } r \geq 0 \end{aligned}$$

which suggests $\lim_{r\to 0} \frac{f(r)}{r} = +\infty$ in the center limit. Together with the finite first order derivatives of metric function

$$\frac{df(r)}{dr} = \text{finite} \quad \text{when } r > 0$$

hence the geodesic curvature $\kappa_g(r)$ in the center limit is

$$\begin{aligned} \lim_{r\to 0} \kappa_g(r) &= \lim_{r\to 0} \left[\frac{1}{\sqrt{f(r)\cdot g(r)}} \left(\frac{f(r)}{r} - \frac{1}{2}\frac{df(r)}{dr} \right) \right] \\ &= \lim_{r\to 0} \left[\frac{1}{\sqrt{f(r)\cdot g(r)}} \Big(+\infty - \text{finite} \Big) \right. \\ &= +\infty \end{aligned} \tag{19}$$

Therefore, for naked singularity spacetime with finite first order metric derivatives, the geodesic curvature satisfies $\lim_{r\to\infty} \kappa_g(r) = 0^+$ at infinite distance limit and $\lim_{r\to 0} \kappa_g(r) = +\infty$ in the center limit, which is precisely what we have depicted in the second case of figure 1, suggesting $\kappa_g(r) = 0$ possesses either no solution or even number of solutions. In such naked singularity spacetimes, the existence of photon spheres is not necessary, and the total number of photon spheres must be even $n = n_{\text{stable}} + n_{\text{unstable}} = 2k$, where $k \in \mathbb{N}$ is a natural number.

Naked Singularity Spacetime (with Divergent First Order Derivatives of Metric): Now we consider an alternative situation for naked singularity spacetime, in which the first order derivatives of metric components become divergent at the spacetime singularity $r = 0$, such that the proper definition of Riemannian curvature tensors is impossible (directly leads to the emergence of spacetime singularity).

Because of the absence of event horizons in naked singularity spacetime, the signs of metric components $f(r)$ and $g(r)$ must be unchanged

$$\begin{aligned} f(r) &= \text{finite} > 0 \quad \text{when } r > 0 \\ g(r) &= \text{finite} > 0 \quad \text{when } r > 0 \end{aligned}$$

which suggests $\lim_{r\to 0} \frac{f(r)}{r} = +\infty$ in the center limit. However, the behavior of geodesic curvature in 2-dimensional optical geometry seems subtle due to the divergent nature of second term in the bracket (which is the metric derivative $\frac{df(r)}{dr}$, see equation (9)). It is necessary to discuss how metric derivative $\frac{df(r)}{dr}$ is divergent at the spacetime singularity $r = 0$.

On one hand, assuming the metric derivative diverges to positive infinity at spacetime singularity ($\lim_{r\to 0} \frac{df(r)}{dr} = +\infty$), the geodesic curvature $\kappa_g(r)$ in the center limit reduces to

$$\begin{aligned} &\lim_{r\to 0} \frac{df(r)}{dr} = +\infty \\ \Rightarrow\ &\lim_{r\to 0} \kappa_g(r) = \lim_{r\to 0} \left[\frac{1}{\sqrt{f(r)\cdot g(r)}} \left(\frac{f(r)}{r} - \frac{1}{2}\frac{df(r)}{dr} \right) \right] \\ &\qquad = \lim_{r\to 0} \left[\frac{1}{\sqrt{f(r)\cdot g(r)}} \Big(+\infty - (+\infty) \Big) \right. \\ &\qquad = \text{indefinite} \end{aligned} \tag{20}$$

In such case, it is impossible to determine whether the equation $\kappa_g(r) = 0$ have solutions or not, as illustrated in the third case of figure 1. Consequently, the existence of photon sphere in this kind of naked singularity spacetime is non-deterministic.

On the other hand, assuming the metric derivative diverges to negative infinity at spacetime singularity ($\lim_{r\to 0} \frac{df(r)}{dr} = -\infty$), the geodesic curvature $\kappa_g(r)$ in the center limit becomes

$$\begin{aligned} &\lim_{r\to 0} \frac{df(r)}{dr} = -\infty \\ \Rightarrow\ &\lim_{r\to 0} \kappa_g(r) = \lim_{r\to 0} \left[\frac{1}{\sqrt{f(r)\cdot g(r)}} \left(\frac{f(r)}{r} - \frac{1}{2}\frac{df(r)}{dr} \right) \right] \\ &\qquad = \lim_{r\to 0} \left[\frac{1}{\sqrt{f(r)\cdot g(r)}} \Big(+\infty - (-\infty) \Big) \right] \\ &\qquad = +\infty \end{aligned} \tag{21}$$

Combing the geodesic curvature $\lim_{r\to\infty} \kappa_g(r) = 0^+$ in the infinite distance limit and $\lim_{r\to 0} \kappa_g(r) = +\infty$ in the center limit, the equation $\kappa_g(r) = 0$ possesses either no solution or even number of solutions, which is precisely what we have depicted in the second case of figure 1. Consequently, this kind of naked singularity spacetime (with metric derivative $\frac{df(r)}{dr}$ diverge to positive infinity

at spacetime singularity) admits even number of photon spheres.

In conclusion, for naked singularity spacetime with divergent first order derivatives of metric, the divergent behavior of metric derivative at spacetime singularity $r=0$ greatly affects the existence of photon sphere. If metric derivative $\frac{df(r)}{dr}$ diverges to positive infinity at spacetime singularity, the existence of photon sphere and total number of photon spheres in naked singularity spacetime are non-deterministic. Conversely, if metric derivative $\frac{df(r)}{dr}$ diverges to negative infinity at spacetime singularity, the photon spheres are not necessarily existed in such naked singularity spacetime, and the total number of photon spheres must be even $n = n_{\text{stable}} + n_{\text{unstable}} = 2k$ (where $k \in \mathbb{N}$ is a natural number).

IV. DISTRIBUTION OF STABLE AND UNSTABLE PHOTON SPHERES

In this section, we present discussions on the distribution properties of stable and unstable photon spheres in different categories of spacetimes. The subtraction of stable and unstable photon spheres $w = n_{\text{stable}} - n_{\text{unstable}}$ is mostly focusing in this section. It is one of the significantly important topics in gravity theories, through which the underlying physical properties of gravitational sources are revealed and tested. Notably, a number of recent studies have show that the distribution of stable and unstable photon spheres can reflect the topological properties of gravitational spacetimes, and the subtraction of unstable photon spheres and stable photon spheres $w = n_{\text{stable}} - n_{\text{unstable}}$ act as a topological invariant / topological charge of spacetime [35, 36, 41, 42]. In this work, instead of using topological approach proposed by Cunha *et al.* and Wei *et al.*, we follow the geometric analysis in reference [60] and give a study based on curvatures in optical geometry.

The most important property for photon sphere distribution is that the stable and untable photon spheres are one-to-one alternatively separated distributed:

> The stable and unstable photon spheres in spacetimes are one-to-one alternatively separated from each other, such that each unstable photon sphere is sandwiched between two stable photon spheres (and each stable photon sphere is sandwiched between two unstable photon spheres).

This distribution characteristics for stable and unstable photon spheres is illustrated in figure 2. Naively speaking, the one-to-one alternatively separated distribution can be understood as the successive appearance of the local maximum and local minimum of effective potentials during the photon motions (where the continuous of effective potential is assumed in literature for a number of gravitational systems). However, this feature is also closely connected with the geometric properties of optical geometry, and it can be proved rigorously by mathematical method. The rigorous demonstration of this one-to-one alternatively separated distribution property can be accomplished through the Gauss-Bonnet theorem, which was first proposed in our recent work [60]. Although the original proof in reference [60] was proposed for black hole spacetimes, the demonstration process is irrelevant to the information of black hole singularity and event horizons, hence it can be applied to other categories of spacetimes (such as ultra-compact object's spacetimes, regular spacetimes and naked singularity spacetimes) without any changes.

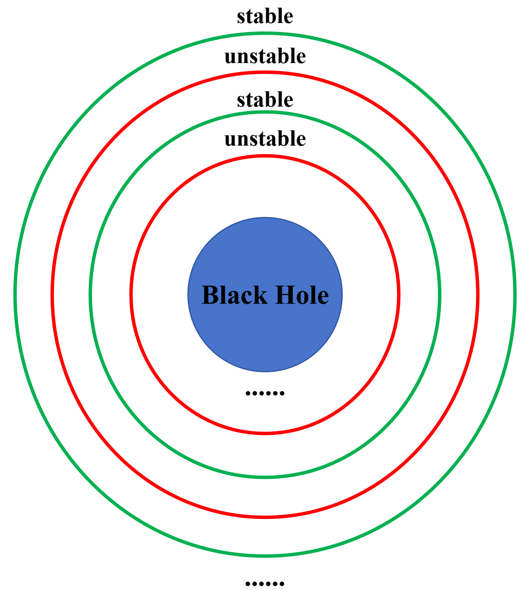

FIG. 2. The stable and unstable photon spheres in spacetimes are one-to-one alternatively separated from each other.

The one-to-one alternatively separated distribution property of stable and unstable photon spheres can easily lead to some nontrivial conclusions, especially constrain the subtraction of stable and unstable photon spheres $w = n_{\text{stable}} - n_{\text{unstable}}$ for different spacetimes and gravitational sources. For ultra compact object's spacetimes, regular spacetimes (without horizons) and naked singularity spacetimes with finite first order derivatives (these spacetimes have even number of photon spheres $n = n_{\text{stable}} + n_{\text{unstable}} = 2k$), the one-to-one alternatively separated distribution forces the numbers of stable and unstable photon sphere equal to each other. In these spacetimes, the subtraction of unstable and stable photon spheres $w = n_{\text{stable}} - n_{\text{unstable}}$, which serves as a topological charge to classify different kinds of spacetime, is vanished for these spacetimes. This successfully recovers a recent theorem raised by Cunha *et al.* [36], which addressed that the topological charge for ultra-compact spacetimes is $w = n_{\text{stable}} - n_{\text{unstable}} = 0$. From our geometric analysis, it is revealed that this relation is not only satisfied for ultra compact object's space-

times, but also hold for a large categories of spacetimes with even numbers of photon spheres, including regular spacetimes without horizons, naked singularity spacetimes with finite first order derivative, naked singularity spacetimes with first order metric derivative $\frac{df(r)}{dr}$ divergent to negative infinity at spacetime singularity ($\lim_{r\to 0}\frac{df(r)}{dr} = -\infty$). The most common characteristic of these spacetimes is the absence of event horizons, in agreement with the Cunha's conclusion in references [36, 61], which suggest the presence and absence of event horizon greatly influenced the topological charge of spacetimes.

On the other hand, for spacetimes admit event horizons, such as black hole spacetimes and regular spacetime (with event horizons), the total number of photon sphere is an odd number $n = n_{\text{stable}} + n_{\text{unstable}} = 2k + 1$. The one-to-one alternatively separated distribution forces the subtraction of stable and unstable photon sphere to be $w = n_{\text{stable}} - n_{\text{unstable}} = \pm 1$. To further constrain the subtraction of stable and unstable photon spheres to be $w = n_{\text{stable}} - n_{\text{unstable}} = -1$, it is necessary to demonstrate that the innermost (or outermost) photon sphere is unstable photon sphere, such that unstable photon spheres in spacetime cannot be less than stable photon spheres. Conversely, to constrain the subtraction of stable and unstable photon spheres to be $w = n_{\text{stable}} - n_{\text{unstable}} = +1$, it is necessary to show that the innermost (or outermost) photon sphere is stable photon sphere. Fortunately, our recent work show that the innermost photon sphere in black hole spacetimes (which is closet to event horizon) must be unstable photon sphere [60]. In principle, the analysis is valid for any spherically symmetric spacetime with event horizons (such as regular spacetime with event horizons), not only restricted to black hole spacetimes. In this way, for black hole spacetimes and regular spacetime with event horizons, the subtraction of unstable and stable photon sphere (serves as a topological charge of spacetime) must be $w = n_{\text{stable}} - n_{\text{unstable}} = -1$, rather than $w = n_{\text{stable}} - n_{\text{unstable}} = +1$. This result successfully recovers a recent theorem on topological charge in black hole spacetimes [36, 42, 61].

However, for naked singularity spacetimes with first order metric derivative $\frac{df(r)}{dr}$ divergent to positive infinity at spacetime singularity ($\lim_{r\to 0}\frac{df(r)}{dr} = +\infty$), the existence and total number of photon spheres are both non-deterministic, then the one-to-one alternatively separated distribution of stable and unstable photon spheres cannot constrain the subtraction of stable and unstable photon sphere $w = n_{\text{stable}} - n_{\text{unstable}}$ to be a definite value.

V. CONCLUSIONS AND PROSPECTS

The photon sphere is a significantly important topic in the studies of black holes and other astrophysical objects. It not only closely connected with the astrophysical observations, but also unveils the underlying physics of various gravitational sources. The number and distributions of photon spheres are strongly influenced by the features of gravitational fields, physical properties of gravitational sources, geometric and topological properties of spacetimes. In different categories of spacetimes (such as black hole spacetime, ultra-compact object's spacetimes, regular spacetimes, naked singularity spacetimes), the existence, number and distribution of photon spheres could exhibit entirely different features.

In this study, we present a general discussion on photon spheres for different categories of spacetimes, based on geometric curvatures and geometric properties of optical geometry. Though a geometric analysis, we successfully derive conclusions on existence of photon spheres, total number of photon spheres $n = n_{\text{stable}} + n_{\text{unstable}}$, the subtraction of stable and unstable photon sphere numbers $w = n_{\text{stable}} - n_{\text{unstable}}$ (which can be viewed as a topological charge to clarify different categories of spacetime) in black hole spacetimes, ultra-compact objects' spacetimes, regular spacetime (with the presence and absence of event horizons), naked singularie spacetimes (with finite or divergent first order derivative of metric). The detailed conclusions on photon spheres obtained in the present work are summarized in table II.

Assuming the most general asymptotic behaviors of spacetimes (asymptotically flat, asymptotically de-Sitter and asymptotically anti de-Sitter), the photon spheres in different categories of spacetimes can be summarized:

- **Black Hole Spacetime:** At least one photon spheres are existed in black hole spacetime (outside the event horizon). The total number of photon sphere in black hole spacetime is an odd number $n = n_{\text{stable}} + n_{\text{unstable}} = 2k + 1$, and the subtraction of stable and unstable photon sphere fulfills $w = n_{\text{stable}} - n_{\text{unstable}} = -1$.

- **Compact Object's Spacetime:** The photon sphere is not necessarily existed in ultra-compact object's spacetime. If photon spheres exist near in ultra compact object's spacetime, their total number must be an even number $n = n_{\text{stable}} + n_{\text{unstable}} = 2k$. The subtraction of stable and unstable photon spheres satisfies $w = n_{\text{stable}} - n_{\text{unstable}} = 0$.

- **Regular Spacetime (With Horizon):** The photon spheres in regular spacetime (with the presence of horizons) are similar to those in black hole spacetime. There are at least one photon spheres existed in regular spacetime (outside the event horizon). The total number of photon spheres is an odd number $n = n_{\text{stable}} + n_{\text{unstable}} = 2k + 1$, and the subtraction of stable and unstable photon spheres must be $w = n_{\text{stable}} - n_{\text{unstable}} = -1$.

- **Regular Spacetime (Without Horizon):** The photon spheres in regular spacetime (without the

TABLE II. Features of photon spheres in different categories of spacetimes, which are obtained using our geometric analysis.

Category of Spacetime	Existence of Photon Spheres	Distribution of Photon Spheres $n = n_{\text{stable}} + n_{\text{unstable}}$	$w = n_{\text{stable}} - n_{\text{unstable}}$	Unstable and Stable Photon Sphere Number	
Black Hole Spacetime	Must Be Existed	$n = 2k+1$	$w = -1$	$n_{\text{unstable}} = k+1$	$n_{\text{stable}} = k$
Ultra-Compact Object's Spacetime	Not Necessarily Existed	$n = 2k$	$w = 0$	$n_{\text{unstable}} = k$	$n_{\text{stable}} = k$
Regular Spacetime (With Horizon)	Must Be Existed	$n = 2k+1$	$w = -1$	$n_{\text{unstable}} = k+1$	$n_{\text{stable}} = k$
Regular Spacetime (Without Horizon)	Not Necessarily Existed	$n = 2k$	$w = 0$	$n_{\text{unstable}} = k$	$n_{\text{stable}} = k$
Naked Singularity Spacetime (with Finite $\frac{df(r)}{dr}$)	Not Necessarily Existed	$n = 2k$	$w = 0$	$n_{\text{unstable}} = k$	$n_{\text{stable}} = k$
Naked Singularity Spacetime (with $\lim_{r\to 0}\frac{df(r)}{dr} = +\infty$)	Non-deterministic	Non-deterministic	Non-deterministic	Non-deterministic	
Naked Singularity Spacetime (with $\lim_{r\to 0}\frac{df(r)}{dr} = -\infty$)	Not Necessarily Existed	$n = 2k$	$w = 0$	$n_{\text{unstable}} = k$	$n_{\text{stable}} = k$

presence of horizons) are similar to those in ultra-compact object's spacetime, where photon spheres are not necessarily existed in such spacetime. If photon spheres exist, the total number of photon spheres in regular spacetime (without horizons) must be an even number $n = n_{\text{stable}} + n_{\text{unstable}} = 2k$, and the subtraction of stable and unstable photon spheres is $w = n_{\text{stable}} - n_{\text{unstable}} = 0$.

- **Naked Singularity Spacetime (with Finite First Order Metric Derivative):** Assuming all first order metric derivatives are finite at the spacetime singularity $r = 0$, the photon sphere is not necessarily existed in this kind of naked singularity spacetime. If photon spheres exist, their total number must be even $n = n_{\text{stable}} + n_{\text{unstable}} = 2k$, and the subtraction of stable and unstable photon spheres is $w = n_{\text{stable}} - n_{\text{unstable}} = 0$.

- **Naked Singularity Spacetime (with Divergent First Order Metric Derivative at Spacetime Singularity)**: In this kind of spacetime, the divergent behavior of metric derivative at spacetime singularity greatly influence the existence and distribution of photon spheres. Firstly, if metric derivative $\frac{df(r)}{dr}$ diverges to positive infinity at spacetime singularity, the existence and total number of photon spheres in naked singularity spacetime are both non-deterministic. Secondly, if metric derivative $\frac{df(r)}{dr}$ diverges to negative infinity at spacetime singularity, the photon spheres is not necessarily existed in such naked singularity spacetimes. If photon spheres exist, the total number of photon spheres must be an even number $n = n_{\text{stable}} + n_{\text{unstable}} = 2k$, and the subtraction of stable and unstable photon spheres satisfies $w = n_{\text{stable}} - n_{\text{unstable}} = 0$.

Our analysis in the present study is valid in any spherically symmetric (static) gravitational systems with the general spacetime metric $ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2$, regardless of the detailed expressions of metric components $g_{tt} = -f(r)$, $g_{rr} = g(r)$. These conclusions reflect the general geometric and topological properties in different categories of spacetimes, only relying on the most general asymptotic behaviors of spacetimes (asymptotically flat, asymptotically de-Sitter and asymptotically anti de-Sitter), hence they can be safely used in any specific gravitational systems.

Furthermore, our present work also has the ability to stimulate some interesting studies, such as the generalization of our analysis into circular orbits of massive particles. Particularly, the existence of innermost stable circular orbits (ISCO) and the total number of marginally stable circular orbits (MSCO) for various kind of massive particles in spacetimes are significantly important topics in recent years, for their influences on the astrophysical accretion processed and gravitational waves. To apply a similar analyze on circular orbits of massive particles, the geodesic curvature and Gaussian curvature in the Jacobi geometry of spacetimes must be utilized. The further development of our analysis into massive particles circular orbits (such as ISCOs and MSCOs for massive particles) deserve more detailed subsequent studies.

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[91] Strictly speaking, due to the opacity of astrophysical ultra-compact objects (such as dwarf stars and neutron stars), only the photon spheres with radius larger than ultra-compact object's size $r_{ph} \geq r_{\rm surface}$ can be detected by distant observers in astrophysical observations. The photon spheres inside the astrophysical ultra-compact objects are much too complicated, which is beyond the scope of this study. In the current work, we simply assume the size of astrophysical ultra-compact objects to be sufficiently small. Under such circumstances, any photon spheres emerged in the entire spacetime region $r \geq 0$ can be observed.

[92] The geodesic curvature depends on the first-order derivatives of the metric components $f(r)$ and $g(r)$. The continuity of first-order derivatives of metric components $f(r)$, $g(r)$ is necessary to give the well-defined concepts of curvature tensors (or curvature scalars), both in 4-dimensional spacetime geometry and the corresponding 2-dimensional optical geometry. For spacetimes admit event horizons, their optical geometry are generally defined outside the event horizon $r \geq r_H$, in which the spacetime singularities (of black holes) are excluded. So we always assume the continuity of geodesic curvature in optical geometry, expect for the center point $r = 0$ of the horizon-less naked singularities spacetime.